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The motion of a charged particle in general relativity

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Abstract. A new approach to the problem of the motion of a self-interacting massive charged particle in general relativity is presented. A charged Robinson-Trautman solution is used as a general relativistic model of such a particle. Such a solution is shown to generate a unique world line in its own H space, which is interpreted as the world line of the particle. Using the R-T dynamical relations, the equation of motion of the particle is derived, which, in the limiting case of zero curvature, is shown to be the same as the classical Lorentz-Dirac equation of motion.

1. Introduction

The purpose of this paper is to introduce a new approach to the problem of motion in general relativity which makes essential use of the H space of Newman and Penrose and others (see e.g. Hallidy and Ludvigsen 1979, Hansen and Ludvigsen 1977, Hansen *et al* 1978, Ko *et al* 1976, 1977, Lind *et al* 1972, Ludvigsen 1977, 1978). In this paper we consider only the motion of a charged self-interacting particle. The motion of a general system will be discussed in a later paper. (Hallidy and Ludvigsen 1979, Ko *et al* 1977).

A charged Robinson-Trautman solution (see Newman and Posadas 1969) is used as a general relativistic model of a classical self-interacting charged particle. Such solutions possess the unique property that the null cones emanating from the singularity are completely shear-free. These null cones intersect $\mathscr{I}^+$ in a one-parameter family of shear-free cuts and, when $\mathscr{I}^+$ is complex, describe a world line in the H space of the solution. This world line is interpreted as the world line of the particle. In the case of flat space-time M with the singularity replaced by a particle, H can be identified with M, and the world line in H can be identified with the world line of the particle†. In the case of curved space-time, however, H cannot be identified with the original space-time; but the world line in H can be thought of as, in a sense, the asymptotically observed world line of the particle.

Conserved quantities m and e are obtained from the dynamical equations governing the solution and are defined to be the mass and charge of the solution. A Maxwell field $\phi_{A'B'}$ on H is obtained from the radiation component $\tilde{\phi}_2^0$ of the Maxwell field of the solution. By comparison with the flat space case $\phi_{A'B'}$ is interpreted as the radiation field of the particle. Making use of the dynamical equations (essentially the Bianchi identities), the equation of motion of the particle is obtained and shown to have the

† Strictly speaking, the H space of flat space-time M should be identified with a complex thickening $C'M$ of M. However, since, in the case of flat space-time, the H space construction works for an arbitrarily small thickening, we may take the thickening to be zero and identify H with M (see footnote on page 1758).

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form

$$m\dot{v}^{AA'} = -\sqrt{2}\, e v^{A}_{B'} \phi^{A'B'},$$

which, in the limiting case of zero curvature, is shown to be the same as the Lorentz–Dirac equation of motion for a charged particle. Unlike the classical equation, this equation is obtained without resorting to any renormalisation procedure or *ad hoc* assumptions.

Sections 2 and 3 are intended as a review of the properties of weighted functions on $C'\mathscr{I}^+$ and of the intrinsic H-space formalism. Since most of this material has been covered elsewhere (Hansen and Ludvigsen 1977), few detailed calculations are given.

In § 4 the radiation component of the Maxwell field of an Einstein–Maxwell solution is shown to determine a Maxwell field on the H space of the solution. This field is interpreted as the radiation field of the solution. In § 5 the classical Lorentz–Dirac equation of motion for a charged particle is discussed; and finally, in § 6 the general relativistic equation of motion for a charged particle is derived This is shown to have the same form as the corresponding equation of § 5.

It is assumed that the reader is familiar with the properties of $\mathscr{I}^+$ as well as with spin-weighted functions and the associated ð (edth) operator.

2. *H* space and weighted functions on $C'\mathscr{I}^+$

In this section we present a review of the intrinsic H space formalism and weighted functions on $C'\mathscr{I}^+$. The reader is referred to Hansen and Ludvigsen (1977), Ko *et al* (1977) and Ludvigsen (1977) for a fuller discussion of these topics.

Let M be the space–time of an asymptotically flat solution of the Einstein–Maxwell equations and $\mathscr{I}^+$ (Penrose 1968) its future null infinity, defined by $\Omega = 0$, where Ω is the conformal factor. In general there are no good cuts (shear-free space-like cross sections) of $\mathscr{I}^+$. However, if $\mathscr{I}^+$ is extended into the complex by allowing the coordinates on $\mathscr{I}^+$ to assume complex values, good cuts can be shown to exist and to form a four-dimensional complex manifold (Hansen *et al* 1978). The resulting complex $\mathscr{I}^+$ will be denoted by $C\mathscr{I}^+$. Actually, we do not need the whole of $C\mathscr{I}^+$ but only some sufficiently large complex thickening $C'\mathscr{I}^+$ of $\mathscr{I}^+$, where

$$\mathscr{I}^+ \subset C'\mathscr{I}^+ \subset C\mathscr{I}^+.$$

We assume that $C'\mathscr{I}^+$ can be chosen such that it is large enough to allow a four-complex-parameter family of good cuts, and yet small enough to ensure that all physically interesting functions, which are assumed analytic on $\mathscr{I}^+$, are holomorphic on $C'\mathscr{I}^+$ †. As we shall see, the four-manifold of good cuts can be endowed with a naturally defined Riemannian metric. Newman and Penrose and others call this space H space.

Let $\{x^a\}$ be some coordinate system on H, and $x^a = x^a(u)$ some world line in H, where u is a complex parameter. To each value of u there is a point of H and hence a good cut of $C'\mathscr{I}^+$. u may therefore be used as a parameter along the generators of $C'\mathscr{I}^+$. Labelling the generators of $C'\mathscr{I}^+$ by stereographic coordinates ζ, $\tilde{\zeta}$, we see that $(u, \zeta, \tilde{\zeta})$ forms a coordinate system on $C'\mathscr{I}^+$. Note that, since we are in the complex domain, $\tilde{\zeta}$ is not restricted to be the complex conjugate of ζ. Each u = constant cut of $C'\mathscr{I}^+$ defines

† Throughout this paper we choose the complex thickening of $\mathscr{I}^+$ to be as small as possible. In particular, in the case of flat space–time, we choose it to be zero and identify $C'\mathscr{I}^+$ with $\mathscr{I}^+$; we then have $H = M$.

an asymptotically shear-free null hypersurface in $C'M$, where $C'M$ is the complex thickening of M.

By introducing r, a parameter along the generators of these hypersurfaces, normalised such that

$$r = \Omega + O(\Omega^2),$$

it can be seen that

$$\{x^a\} = (u, r, \zeta, \tilde{\zeta})\ \dagger$$

can be used as a coordinate system in the neighbourhood of $C'\mathscr{I}^+$. A vector $\hat{n}^a$, tangent to the generators of $C'\mathscr{I}^+$, is defined by

$$\hat{n}^a = dx^a/du = \delta^a_0.$$

Since $C'\mathscr{I}^+$ is null and Ω vanishes on $C'\mathscr{I}^+$, $\Omega_{,\mathbf{a}}$ is also tangent to the generators of $C'\mathscr{I}^+$. We therefore choose Ω such that

$$\Omega_{,\mathbf{a}} = -\hat{n}_{\mathbf{a}} \quad \text{on } C'\mathscr{I}^+.$$

In terms of this system, the line element on $C'\mathscr{I}^+$ assumes the form

$$ds^2 = -(2\, du\, dr + d\zeta\, d\tilde{\zeta}/2P^2). \tag{2.1}$$

An associated null tetrad $(\hat{l}^a \hat{n}^a \hat{m}^a \hat{\tilde{m}}^a)$ on $C'\mathscr{I}^+$ is given by

$$\hat{l}^a = -\hat{\delta}^a_1, \qquad \hat{n}^a = \delta^a_0, \qquad \hat{m}^a = -2P\delta^a_2, \qquad \hat{\tilde{m}}^a = -2P\delta^a_3.$$

Using (2.1) it is easily checked that these vectors do in fact satisfy the usual null tetrad orthogonality relations. It is convenient to express the null tetrad in terms of two spinor dyads $(\hat{o}_{\mathbf{A}}, \hat{\iota}_{\mathbf{A}})$ and $(\hat{o}_{\mathbf{A}'}, \hat{\iota}_{\mathbf{A}'})$, where

$$\hat{o}_{\mathbf{A}}\hat{\iota}^{\mathbf{A}} = \hat{o}_{\mathbf{A}'}\hat{\iota}^{\mathbf{A}'} = 1$$

and

$$\hat{n}^a = \hat{\iota}^{\mathbf{A}}\hat{\iota}^{\mathbf{A}'}, \qquad \hat{l}^a = \hat{o}^{\mathbf{A}}\hat{o}^{\mathbf{A}'}, \qquad \hat{m}^a = \hat{o}^{\mathbf{A}}\hat{\iota}^{\mathbf{A}'}, \qquad \hat{\tilde{m}}^a = \hat{\iota}^{\mathbf{A}}\hat{o}^{\mathbf{A}'}.$$

Let $\psi_{\mathbf{ABCD}}$ and $\phi_{\mathbf{AB}}$ be the Weyl and Maxwell spinors associated with the solution. Under the condition of asymptotic flatness, it can be shown that $\psi_{\mathbf{ABCD}}\Omega^{-1}$ and $\phi_{\mathbf{AB}}\Omega^{-1}$ are finite and smooth on $C'\mathscr{I}^+$ (Penrose 1968). We may therefore define the following components of these quantities on $C'\mathscr{I}^+$:

$$\psi^0_0 = \Omega^{-1}\psi_{\mathbf{ABCD}}\hat{o}^{\mathbf{A}}\hat{o}^{\mathbf{B}}\hat{o}^{\mathbf{C}}\hat{o}^{\mathbf{D}}, \qquad \psi^0_1 = \Omega^{-1}\psi_{\mathbf{ABCD}}\hat{o}^{\mathbf{A}}\hat{o}^{\mathbf{B}}\hat{o}^{\mathbf{C}}\hat{\iota}^{\mathbf{D}},$$

$$\psi^0_2 = \Omega^{-1}\psi_{\mathbf{ABCD}}\hat{o}^{\mathbf{A}}\hat{o}^{\mathbf{B}}\hat{\iota}^{\mathbf{C}}\hat{\iota}^{\mathbf{D}}, \qquad \psi^0_3 = \Omega^{-1}\psi_{\mathbf{ABCD}}\hat{o}^{\mathbf{A}}\hat{\iota}^{\mathbf{B}}\hat{\iota}^{\mathbf{C}}\hat{\iota}^{\mathbf{D}},$$

$$\psi^0_4 = \Omega^{-1}\psi_{\mathbf{ABCD}}\hat{\iota}^{\mathbf{A}}\hat{\iota}^{\mathbf{B}}\hat{\iota}^{\mathbf{C}}\hat{\iota}^{\mathbf{D}}, \qquad \phi^0_0 = \Omega^{-1}\phi_{\mathbf{AB}}\hat{o}^{\mathbf{A}}\hat{o}^{\mathbf{B}},$$

$$\phi^0_1 = \Omega^{-1}\phi_{\mathbf{AB}}\hat{o}^{\mathbf{A}}\hat{\iota}^{\mathbf{B}}, \qquad \phi^0_2 = \Omega^{-1}\phi_{\mathbf{AB}}\hat{\iota}^{\mathbf{A}}\hat{\iota}^{\mathbf{B}},$$

and similarly for $\tilde{\phi}^0_i$ and $\tilde{\psi}^0_i$.

Let $x^a_0 = x^a(u_0)$ be some point on the world line. The cut associated with a neighbouring point $x^a_0 + dx^a$ must have the form

$$u = u_0 + L_a(u_0, \zeta, \tilde{\zeta})\, dx^a.$$

† Throughout this paper we use the convention that objects with bold indices live in M, while objects with ordinary indices live in H.

This defines the quantity L_a. It can be shown that L_a must satisfy (Hallidy and Ludvigsen 1979, Luuvigsen 1977)

$$\eth^2 L_a = 0. \tag{2.2}$$

Here $\eth$, the generalised edth operator associated with P, is defined by

$$\eth\eta = 2P^{1-s}\,\partial(P^s\eta)/\partial\zeta, \tag{2.3a}$$

where η has spin weight s. Similarly $\tilde{\eth}$ is defined by

$$\tilde{\eth}\eta = 2P^{1+s}\,\partial(P^{-s}\eta)/\partial\tilde{\zeta}. \tag{2.3b}$$

If $x^a(u)$ and $x^a(u+\mathrm{d}u) = x^a + \mathrm{d}x^a$ are two neighbouring points on the world line, then

$$\mathrm{d}u = L_a\,\mathrm{d}x^a.$$

Thus

$$L_a v^a = 1, \tag{2.4}$$

where v^a is defined to be $\dot{x}^a (= \mathrm{d}x^a/\mathrm{d}u)$.

The (complex) surface element of the (u = constant) cuts of $C'\mathscr{I}^+$ is given by $\mathrm{d}\zeta \wedge \mathrm{d}\tilde{\zeta}/2\mathrm{i}P^2$, and we define $\mathrm{d}S (= (1/4\pi) \times$ surface element) by

$$\mathrm{d}S = \mathrm{d}\zeta \wedge \mathrm{d}\tilde{\zeta}/8\pi i \mathrm{P}^2.$$

By the above prescription we have been able to define P, ϕ_i^0, $\tilde{\phi}_i^0$, ψ_i^0, $\tilde{\psi}_i^0$, L_a and $\mathrm{d}S$ uniquely on each (u = constant) cut, and hence on each point of the world line. By employing a space-filling congruence of such world lines or, equivalently, a vector field v^a tangent to the congruence, these quantities can be defined at all points of H. They have the functional form $\eta(x^a, \zeta, \tilde{\zeta}, v^a)$. Using a different congruence and hence a different v^a field, these quantities can again be found. It can be shown that they transform according to

$$P(v^{a'}) = WP(v^a), \qquad \phi_i^0(v^{a'}) = W^2\phi_i^0(v^a), \qquad \tilde{\phi}_i^0(v^{a'}) = W^2\tilde{\phi}_i^0(v^a),$$
$$\psi_i^0(v^{a'}) = W^3\psi_i^0(v^a), \qquad \tilde{\psi}_i^0(v^{a'}) = W^3\tilde{\psi}_i^0(v^a), \qquad L_a(v^{a'}) = W^{-1}L_a(v^a),$$
$$\mathrm{d}S(v^{a'}) = W^{-2}\,\mathrm{d}S(v^a),$$

where $W = L_a v^{a'}$. This leads us to make the following definition: A quantity $\eta(x^a, \zeta, \tilde{\zeta}, v^a)$ which transforms according to

$$\eta(v^{a'}) = W^{-w}\eta(v^a)$$

will be said to have H conformal weight (HCW) w. Using the fact that $\eth^2 W = 0$, one can prove the following theorem (Hansen and Ludvigsen 1977):

Theorem 1. If η has spin weight (SW) s and HCW w, where $w \geqslant s$, then $\eth^{w-s+1}\eta$ has HCW $s-1$ (and, of course, SW $w+1$); and conversely. Note that this theorem implies that equation (2.2) is H conformally invariant.

A distance function on H is defined by

$$\|\mathrm{d}x^a\| = \left(\int \frac{\mathrm{d}S}{2(L_a\,\mathrm{d}x^a)^2}\right)^{-1}.$$

By considering the weights of L_a and dS, it is seen that $\|dx^a\|$ is invariant. Newman has shown that $\|dx^a\|$ induces a Riemannian metric g_{ab} on H (Newman 1977):

$$\|dx^a\| = g_{ab}\, dx^a\, dx^b = \left(\int \frac{dS}{2(L_a\, dx^a)^2}\right)^{-1}. \tag{2.5}$$

Using (2.4) and (2.5) we have

$$g_{ab}v^a v^b = \left(\int \frac{dS}{2(L_a v^a)^2}\right)^{-1} = \left(\int \frac{dS}{2}\right)^{-1}. \tag{2.6}$$

Throughout the remainder of this paper it will be found convenient to normalise u such that $v^a v_a = 2$. Equation (2.6) now gives

$$\int dS = 1. \tag{2.7}$$

Differentiating (2.6) with respect to v^a, and again using (2.5), we obtain

$$v_a = 2\int L_a\, dS. \tag{2.8}$$

We conclude this section with a discussion of the spinor decomposition of L_a. It can be shown that L_a is null with respect to the H space metric (Hansen and Ludvigsen 1977); it can therefore be written in terms of spinors:

$$L_a = o_A o_{A'}.$$

So as to be consistent with the weight of L_a, o_A is assumed to have SW $\frac{1}{2}$ and HCW $\frac{1}{2}$, and $o_{A'}$, is assumed to have SW $-\frac{1}{2}$ and HCW $\frac{1}{2}$. If, in addition, o_A and $o_{A'}$ satisfy

$$ðo_A = 0 \tag{2.9}$$

and

$$ð^2 o_{A'} = 0, \tag{2.10}$$

then $ð^2 L_a = 0$ is automatically satisfied. Note that theorem 1 guarantees that equations (2.9) and (2.10) are invariant. It may be shown that o_A and $o_{A'}$ can be chosen such that

$$o_A \iota^A = o_{A'}\iota^{A'} = 1,$$

where $\iota_A = \tilde{ð} o_A$ and $\iota_{A'} = ð o_{A'}$ [Hansen and Ludvigsen 1977). Thus

$$\epsilon_{AB} = 2o_{[A}\iota_{B]}. \qquad \epsilon_{A'B'} = 2o_{[A'}\iota_{B']}, \tag{2.11}$$

where

$$g_{ab} = \epsilon_{AB}\epsilon_{A'B'}. \tag{2.12}$$

3. The F function and curvature properties of H

In the previous section the function $P(x^a, \zeta, \tilde{\zeta}, v^a)$ was defined. In terms of P, the Gaussian curvature of the cut corresponding to x^a (in the scaling defined by v^a) is given by

$$K = ð\tilde{ð} \ln P.$$

For flat H space, K is unity, but in general $K-1$ does not vanish. In general it can be shown that K may be written in the form (Hansen and Ludvigsen 1977)

$$K = 1 - \tfrac{1}{2}\eth^2 F. \tag{3.1}$$

Equation (3.1) defines the F function. Using the known transformation law for P, it can be shown that F transforms with HCW 0 and, of course, SW -2. Both F and K will play a major role in the following analysis.

Making use of the well-known formula (Lind *et al* 1972)

$$(\tilde{\eth}\eth - \eth\tilde{\eth})\eta = 2sK\eta, \tag{3.2}$$

where η has SW s, it may be shown that

$$\tilde{\eth} o_{A'} = -\tfrac{1}{2}\eth F o_{A'} + F\iota_{A'}. \tag{3.3}$$

Contracting (3.3) with $o^{A'}$, we obtain

$$o^{A'}\tilde{\eth} o_{A'} = -F. \tag{3.4}$$

Using (2.4), (2.12) and (3.4) it can be shown that

$$v_{AA'} = o_A o_{A'} + \iota_A \iota_{A'} - \tfrac{1}{2}\eth F o_A \iota_{A'}. \tag{3.5}$$

Contracting (3.5) with $\iota^{A'}$ we obtain

$$v_{AA'}\iota^{A'} = o_A. \tag{3.6}$$

Let ∇_a be the covariant derivative on H. Since ∇_a does not in general preserve HCW (that is, if η has a well-defined HCW, $\nabla_a \eta$ in general does not), we introduce another differential operator, $þ'_a$, defined by (Hansen and Ludvigsen 1977)

$$þ'_a \eta = (\nabla_a + w(P_a/P))\eta,$$

where w is the HCW of η, and P_a is given by $P_a = \nabla_a P$. One can easily show that $þ'_a$ preserves the HCW of η.

Since, for fixed ζ and $\tilde{\zeta}$, L_a can be considered as being a vector field on H, we may calculate

$$þ'_a L_b = \nabla_a L_b + (P_a/P)L_b.$$

It can be shown that $þ'_a L_b$ is given by (Hansen and Ludvigsen 1977)

$$þ'_a L_b = \tfrac{1}{6}\eth^2 G L_a L_b - \tfrac{2}{3}\eth G L_{(a}\, \eth L_{b)} + G\, \eth L_a\, \eth L_b, \tag{3.7}$$

where

$$G = L^a(þ'_a F) = L^a(\nabla_a F)\dagger.$$

It can also be shown that

$$\eth^2(P_a/P) = \eth^2(\dot{P}/P)L_a \tag{3.8}$$

and

$$\eth^2(\dot{P}/P) = \tfrac{1}{6}\eth^4 G, \tag{3.9}$$

where $\dot{P} = v^a \nabla_a P$. In terms of spinors, equation (3.7) becomes

$$þ'_{AA'} o_B = 0 \tag{3.10}$$

† In Hansen and Ludvigsen (1977) the letter C was used for the G function.

and

$$\text{Þ}_{AA'}o_{B'} = o_A(\tfrac{1}{6}\,\eth G o_A o_{B'} - \tfrac{2}{3}\,\eth G o_{(A'}\iota_{B')} + G\iota_{A'}\iota_{B'}). \tag{3.11}$$

Making use of (3.10) and (3.11), the spinor components of the curvature tensor of H can be found (Hansen and Ludvigsen 1977):

$$\Phi_{ABA'B'} = \Lambda = \Psi_{ABCD} = 0$$

$$\psi_{A'B'C'D'} = \frac{1}{4!}\,\eth^4 H o_{A'}o_{B'}o_{C'}o_{D'} - \frac{1}{3!}\,\eth^3 H o_{(A'}o_{B'}o_{C'}\iota_{D')} + \frac{1}{2!}\,\eth^2 H o_{(A'}o_{B'}\iota_{C'}\iota_{D')}$$
$$-\frac{1}{1!}\eth H o_{(A'}\iota_{B'}\iota_{C'}\iota_{D')} + H\iota_{A'}\iota_{B'}\iota_{C'}\iota_{D'}, \tag{3.12}$$

where

$$H = L^a(\text{Þ}'_a G) = L^a L^b \nabla_a \nabla_b F. \tag{3.13}$$

It is thus seen that the F function completely determines the curvature of H.

In terms of the coordinate and tetrad system described in § 2, the spin coefficient equations on $C'\mathscr{I}^+$ imply that

$$\tilde{\psi}_4^0 = -\eth^2(\dot{P}/P)$$

(see equation (2.30) of Lind *et al* (1972). Thus, by equation (3.9), we have

$$\tilde{\psi}_4^0 = -\tfrac{1}{6}\,\eth^4 G. \tag{3.14}$$

Therefore $\tilde{\psi}_4^0$ may also be used to determine the curvature of H. Comparing equation (3.12) and (3.14) with the spin-2 case in the Appendix, it is seen that $\psi_{A'B'C'D'}$ is obtained from $\tilde{\psi}_4^0$ in the same manner as in a spin-2 field $\eta_{A'B'C'D'}$ is obtained from its null datum η_4^0 on $\mathscr{I}^+$.

We conclude this section by giving a relation between $\dot{P}/P$ and the 'acceleration' $\dot{v}_a(=v^b\nabla_b v_a)$ of the world line (Ludvigsen 1977):

$$\dot{v}_a = -6\int(\dot{P}/P)L_a\,\mathrm{d}S. \tag{3.15}$$

Equation (3.15) may be obtained by taking the covariant derivative of both sides of

$$v_a = 2\int L_a\,\mathrm{d}S$$

and observing that

$$\text{Þ}'_{[a}L_{b]} = 0$$

and

$$\text{Þ}'_a\,\mathrm{d}S = \nabla_a\,\mathrm{d}S + 2(P_a/P)\,\mathrm{d}S = 0. \tag{3.16}$$

4. Maxwell fields on H

In the previous section we saw that $\tilde{\psi}_4^0$ determines the Weyl spinor $\psi_{A'B'C'D'}$ of H in exactly the same way as the null datum η_4^0 of a spin-2 field determines $\eta_{A'B'C'D'}$. It is therefore reasonable to conjecture that the analogous process might work for $\tilde{\phi}_2^0$, giving a Maxwell spinor on H. In this section we show that this conjecture is indeed justified.

By analogy with equation (A2), we define a function B by

$$\tilde{\phi}_2^0 = -\eth^2 B. \tag{4.1}$$

Since $\tilde{\phi}_2^0$ has SW 1 and HCW -2, theorem 1 implies that B has SW -1 and HCW 0. By calculating $\tilde{\phi}_2^0$ at two neighbouring points of H, it can be shown that it satisfies

$$þ'_a \tilde{\phi}_2^0 = L_a þ' \tilde{\phi}_2^0,$$

or, on using (4.1),

$$þ'_a(\eth^2 B) = L_a þ'(\eth^2 B), \tag{4.2}$$

where $þ' = v^a þ'_a$. After a short calculation, using equations (2.3) and (3.8), it can be shown that (4.2) is equivalent to

$$\eth^2(\nabla_a B) = L_a\, \eth^2 \dot{B}. \tag{4.3}$$

Introducing a function C defined by

$$C = L^a \nabla_a B,$$

we see that

$$\eth^3 C = L^a\, \eth^3(\nabla_a B) + 3\, \eth L^a\, \eth^2(\nabla_a B). \tag{4.4}$$

Since $L^a L_b = \eth L^a L_a = 0$, one can see immediately that equations (4.3) and (4.4) imply that $\eth^3 C$ vanishes. By analogy with the Appendix we define a spinor field $\phi_{A'B'}$ by

$$\phi_{A'B'} = \frac{\eth^2 C}{2!} o_{A'} o_{B'} - \frac{\eth C}{1!} o_{(A'}\iota_{B')} + C \iota_{A'} \iota_{B'}. \tag{4.5}$$

$\phi_{A'B'}$ has SW 0 and, after a short calculation, can be shown to have HCW 0. It is therefore independent of the v^a field. Since $\eth^3 C = 0$ and $\eth^2 o_{A'} = \eth \iota_{A'} = 0$, one can easily check that $\eth \phi_{A'B'} = 0$. Thus $\phi_{A'B'}$ is independent of ζ; also, since all quantities are assumed to be holomorphic, $\phi_{A'B'}$ must also be independent of $\tilde{\zeta}$. $\phi_{A'B'}(x^a)$ is therefore a well-defined spinor field on H.

After a rather long calculation, which makes use of (4.3), one can show that $\phi_{A'B'}$ is a Maxwell spinor. That is, it satisfies

$$\nabla_A^{A'} \phi_{A'B'} = 0.$$

It will be found useful to have $\phi_{A'B'}$ expressed in the form of an integral. This can be achieved as follows. Since $\int \mathrm{d}S = 1$, we have

$$\begin{aligned}
\phi_{A'B'} &= \int \phi_{A'B'}\, \mathrm{d}S \\
&= \int \left(\frac{\eth^2 C}{2} o_{A'} o_{B'} - \eth C o_{(A'}\iota_{B')} + C \iota_{A'} \iota_{B'} \right) \mathrm{d}S \\
&= \int \left(\frac{\eth^2 C}{2} o_{A'} o_{B'} - \frac{\eth C}{2} \eth(o_{A'} o_{B'}) + \frac{C}{2} \eth^2(o_{A'} o_{B'}) \right) \mathrm{d}S.
\end{aligned}$$

By integrating by parts (Ludvigsen 1977), this last expression gives

$$\phi_{A'B'} = \tfrac{3}{2} \int \eth^2 C o_{A'} o_{B'}\, \mathrm{d}S. \tag{4.6}$$

Using (4.3) one can show that $\eth^2 C$ can be written in the form

$$\tfrac{3}{2}\eth^2 C = \eth^2 \dot{B} + \eth^3 \chi \tag{4.7}$$

for some function (which need not concern us here) χ. Substituting (4.7) into (4.6), integrating by parts, and using the fact that $\eth^3(o_{A'}o_{B'})$ vanishes, we obtain

$$\phi_{A'B'} = \int \eth^2 \dot{B} o_{A'} o_{B'}\, \mathrm{d}S = \int \dot{B}\, \eth^2(o_{A'}o_{B'})\, \mathrm{d}S = 2\int \dot{B}\iota_{A'}\iota_{B'}\, \mathrm{d}S. \tag{4.8}$$

Finally, contracting with $v_A^{B'}$ and using equation (3.6), we obtain

$$\phi_{A'B'}v_A^{B'} = -2\int \dot{B} o_A \iota_{A'}\, \mathrm{d}S = -2\int \dot{B}\, \eth L_a\, \mathrm{d}S. \tag{4.9}$$

Equation (4.8) will play a central role when we come to consider the equation of motion of a charged particle.

5. The motion of a charged particle in flat space–time

In this section we consider the classical Lorentz–Dirac equation of motion for a charged point particle in flat space–time M.

Since the $\mathscr{I}^+$ of flat space–time possesses a four-real-parameter family of good cuts (namely those cuts formed by the intersection of $\mathscr{I}^+$ by the null cones emanating from points of M), it is not necessary to thicken $\mathscr{I}^+$ into the complex†. We may therefore choose $C\mathscr{I}^+$ to be $\mathscr{I}^+$, and the corresponding H space can be seen to be real and equivalent to the original space–time M. The same index labels may therefore be used for both M and H.

Consider a point particle with charge e, whose world line is given by $x^a = x^a(u)$, where, as usual, u is normalised such that $v_a v^a = 2(v_a = \dot{x}_a)$. Let $\overset{+}{F}_{ab}$ be the retarded field and $\overset{-}{F}_{ab}$ the advanced field produced by the particle. Both $\overset{+}{F}_{ab}$ and $\overset{-}{F}_{ab}$ are obviously singular on the world line. The radiation field $\underset{\mathrm{R}}{F}{}^{ab}$, defined by

$$\underset{\mathrm{R}}{F}{}_{ab} = \overset{+}{F}_{ab} - \overset{-}{F}_{ab},$$

is, however, perfectly regular everywhere and on the world line can be shown to have the explicit form (Rohrlich 1965)

$$\tfrac{1}{3}e(v^a \ddot{v}^b - \ddot{v}^a v^b). \tag{5.1}$$

In spinor notation these fields can be written

$$\overset{+}{F}_{ab} = \overset{+}{\phi}_{AB}\epsilon_{A'B'} + \overset{+}{\phi}_{A'B'}\epsilon_{AB},$$

$$\overset{-}{F}_{ab} = \overset{-}{\phi}_{AB}\epsilon_{A'B'} + \overset{-}{\phi}_{A'B'}\epsilon_{AB},$$

$$\underset{\mathrm{R}}{F}{}_{ab} = \underset{\mathrm{R}}{\phi}{}_{AB}\epsilon_{A'B'} + \underset{\mathrm{R}}{\phi}{}_{A'B'}\epsilon_{AB}.$$

Under the physically reasonable assumption that $\tilde{\phi}_2^{0-}$ vanishes on $\mathscr{I}^+$, we see that

$$\tilde{\phi}_2^0 = \tilde{\phi}_2^{0-} = \underset{\mathrm{R}}{\phi}{}_2^0 \quad \text{on } \mathscr{I}^+.$$

† See footnote on page 1758.

Thus $\tilde{\phi}_2^0$ is the null datum for $\underset{\mathrm{R}}{\phi}_{A'B'}$, and the Maxwell field $\phi_{A'B'}$ formed from $\tilde{\phi}_2^0$ (see § 4 and Appendix) will be equal to $\underset{\mathrm{R}}{\phi}_{A'B'}$.

The Dirac–Lorentz equation of motion for the particle can be written in the form (Rohrlich 1965)

$$m\dot{v}^a = (ev^b/\sqrt{2})\underset{\mathrm{R}}{F}{}^a_b, \tag{5.2}$$

where m is the particle's renormalised mass. Substituting (5.1) into (5.2) we obtain

$$\sqrt{2}\, m\dot{v}^a = \tfrac{2}{3}e^2[\ddot{v}^a - (\dot{v}^b\dot{v}_b/2)v^a],$$

where the right-hand side can be recognised as the Abraham–Dirac–Lorentz radiation reaction term. Using equation (5.1) it can be seen that $\underset{\mathrm{R}}{F}{}^{*ab}v_b$ vanishes on the world line, where F^{*ab} is the dual of F^{ab}. Equation (5.2) is therefore equivalent to

$$m\dot{v}^a = (ev_b/\sqrt{2})(\underset{\mathrm{R}}{F}{}^{ba} - \mathrm{i}\underset{\mathrm{R}}{F}{}^{*ba}). \tag{5.3}$$

Finally, using the well-known fact that (Penrose 1968)

$$F^{ba} - \mathrm{i}F^{*ba} = 2\epsilon^{BA}\phi^{A'B'},$$

it is seen that (5.3) may be written as

$$m\dot{v}^{AA'} = (2/\sqrt{2})ev_{BB'}\epsilon^{BA}\phi^{A'B'}\phi^{A'B'}$$

or

$$m\dot{v}^{AA'} = -\sqrt{2}\, ev^A_{B'}\phi^{A'B'}. \tag{5.4}$$

6. The general relativistic equation of motion for a charged particle†

In the previous section we described the motion of a particle by means of its world line. However, in general relativity, where a particle is usually represented by a singularity in space–time, such a description is not appropriate: little sense can be made of the concept of the world line of a singularity in its own space–time. As we shall see, this difficulty can be avoided by describing the motion not in the original space–time but in H space.

We use a charged Robinson–Trautman type II solution (Newman and Posadas 1969) as a general relativistic model for a charged particle. Such solutions have the property that the null cones emanating from the singularity are completely shear-free. These null cones intersect $\mathscr{I}^+$ in a one-parameter family of good cuts and, when $\mathscr{I}^+$ is made complex, form a world line in the space of all good cuts; that is, a world line in the H space of the solution. This world line will be interpreted as the particle's world line.

The equation of motion for the particle can be obtained from the dynamical equations (essentially the Bianchi identities) governing the evolution of the solution. In the coordinate system on $\mathrm{C}'\mathscr{I}^+$ based on the world line, these equations have the form (Newman and Posadas 1969)

$$\dot{\psi}_2^0 - 3(\dot{P}/P)\psi_2^0 = -\eth\tilde{\eth}K + 2\phi_2^0\tilde{\phi}_2^0, \tag{6.1}$$

† This section supplies details of results announced in Ludvigsen (1978).

$$\eth\psi_2^0 = 4\phi_1^0\bar{\phi}_2^0, \tag{6.2}$$

$$\dot{\phi}_1^0 - 2(\dot{P}/P)\phi_1^0 = -\eth\phi_2^0, \tag{6.3}$$

$$\eth\phi_1^0 = 0. \tag{6.4}$$

Equation (6.4) implies that ϕ_1^0 is independent of ζ. Since it is assumed to be holomorphic on $C'\mathscr{I}^+$, it must also be independent of $\bar{\zeta}$. Thus ϕ_1^0 has the form $\phi_1^0(u)$. Since $\int dS = 1$, we have

$$\phi_1^0 = \int \phi_1^0 \, dS.$$

Differentiating with respect to u and using the fact that $(dS)^{\cdot} = -(2\dot{P}/P)\, dS$, we obtain

$$\dot{\phi}_1^0 = \int [\dot{\phi}_1^0 - 2(\dot{P}/P)\phi_1^0]\, dS = -\int \eth\phi_2^0 \, dS = 0,$$

where we have used equation (6.3). Therefore ϕ_1^0 is a constant. For the special case of a Reissner–Nördstrom solution, ϕ_1^0 is equal to the charge. We therefore define ϕ_1^0 to be the charge e of the particle, i.e. $e = \phi_1^0$. Equations (6.3) and (6.2) now become

$$2e\dot{P}/P = \eth\phi_2^0 \tag{6.5}$$

and

$$\eth\psi_2^0 = 4e\bar{\phi}_2^0 = -4e\, \eth^2 B, \tag{6.6}$$

where $\eth^2 B = -\bar{\phi}_2^0$ (see § 4). Equation (6.6) implies that ψ_2^0 has the form

$$\psi_2^0 = -2\sqrt{2}\, m(u) - 4e\, \eth B. \tag{6.7}$$

Equation (6.7) implies that

$$-2\sqrt{2}\, m = \int \psi_2^0 \, dS.$$

Differentiating with respect to u, and using equations (6.1), (6.7) and (6.5), we obtain

$$\begin{aligned}
-2\sqrt{2}\, \dot{m} &= \int \left(\dot{\psi}_2^0 - 2\psi_2^0 \frac{\dot{P}}{P}\right) dS \\
&= \int \left(\dot{\psi}_2^0 - 3\frac{\dot{P}}{P}\psi_2^0 + \frac{\dot{P}}{P}\psi_2^0\right) dS \\
&= \int \left(-\eth\bar{\eth}K - 2\phi_2^0\, \eth^3 B - 2\sqrt{2}\,\frac{\dot{P}}{P} m - 4e\, \eth B \frac{\dot{P}}{P}\right) dS \\
&= -2 \int (\phi_2^0\, \eth^2 B + \eth B\, \eth\phi_2^0)\, dS \\
&= -2 \int \eth(\phi_2^0\, \eth B)\, dS = 0.
\end{aligned}$$

m is therefore a constant. In the special case of a Reissner–Nördstrom solution, m is equal to the mass of the solution. We therefore define m to be the mass of the particle.

Substituting all this information into (6.1), we obtain

$$6\sqrt{2}\, m\dot{P}/P = -\eth\bar{\eth}K + 4e(\eth B)^{\cdot} - 6\, \eth B\, \eth\phi_2^0 - \phi_2^0\, \eth^2 B. \tag{6.8}$$

Using the definition of ð and equation (6.5), it can be seen that

$$(\eth B)^{\cdot} = (\dot{P}/P)\, \eth B - \eth(\dot{P}/P)B + \eth\dot{B} = (1/2e)(\eth\phi_2 B - \eth^2\phi_2 B) + \eth\dot{B}.$$

Substituting this into (6.8), we obtain

$$6\sqrt{2}\, m\dot{P}/P = -\eth\tilde{\eth}K + 4e\, \eth\dot{B} - 2\eth^2(\phi_2^0 B). \tag{6.9}$$

It can be shown that $\eth\tilde{\eth}K$ has the form (Ludvigsen 1977)

$$\eth\tilde{\eth}K = \eth^2 X.$$

Thus equation (6.9) has the form

$$6\sqrt{2}\, m\dot{P}/P = \eth^2 Y + 4e\, \eth\dot{B}. \tag{6.10}$$

Finally, multiplying both sides of (6.10) by L_a, integrating by parts, and using equations (3.15) and (4.9), we obtain

$$\sqrt{2}\, m\dot{v}_a = -4e \int \eth\dot{B}L_a\, \mathrm{d}S = 4e \int \dot{B}\, \eth L_a\, \mathrm{d}S = -2e\phi_{A'B'}v_A^{B'}$$

or, equivalently,

$$m\dot{v}^{AA'} = -\sqrt{2}\, ev_{B'}^{A}\phi^{A'B'}. \tag{6.11}$$

Equation (6.11) is our equation of motion for the charged particle. Referring back to equation (5.4), it can be seen that it is identical in form to the classical equation of motion in flat space–time. However, the motion is now described in the H space of the solution and not in the original space–time. In the limiting case of zero curvature, M is identical to H, and equation (6.11) becomes the classical equation of motion. For vanishing e, it is seen that the particle moves along a geodesic in H.

As was mentioned in § 2, H is necessarily complex for a radiating system, and so equation (6.11) describes the particle's motion not in a real space–time but in a complex space–time. Thus, within our present conceptual framework, it is difficult to fix any precise physical meaning on equation (6.10).

The reader is referred to Hallidy and Ludvigsen (1979), Ko *et al* (1977) and Ludvigsen (1976) for a fuller discussion of this difficulty.

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Appendix. The generalised Kirchoff integral

Let M be flat space–time and $\underbrace{\eta_{A'B'\dots K'}}_{n}$ a spin-$n/2$ field which is singularity-free throughout M. Note that, since M is flat, it can be identified with H. We may therefore use the same index labels for both M and H. In terms of our formalism, the generalised Kirchoff theorem of Penrose (Newman and Penrose 1968) states that

$$\eta_{A'\dots K'}(Q) = -\int \text{Þ}'\eta_n^0 o_A \dots o_{K'}\, \mathrm{d}S, \tag{A1}$$

where

$$\eta_n^0 = \Omega^{-1}\eta_{A'\dots K'}\hat{\iota}^{A'}\dots\hat{\iota}^{K'} \quad \text{on } \mathscr{I}^+,$$

and the integral is performed over the cut of $\mathscr{I}^+$ corresponding to the point Q in M.

Since η_n^0 has SW $n/2$, one can show, by means of an induction argument, that it has the form

$$\eta_n^0 = -[1/(n-1)!]\,\eth^n\alpha, \tag{A2}$$

where the factor $1/(n-1)!$ has been included for notational purposes. Since η_n^0 has HCW $-n/2-1$ and SW n, theorem 1 implies that α has SW $-n$ and HCW $n-1$.

Using the known transformation properties of $\hat{\iota}_A$ and Ω, one can show that η_n^0 satisfies

$$\text{Þ}'_a\eta_n^0 = L_a\text{Þ}'\eta_n^0. \tag{A3}$$

Since H is flat, the F function vanishes, and equations (3.8) and (3.9) give

$$\eth^2(P_a/P) = L_a\,\eth^2(P/P) = 0.$$

Using this fact, one can show that equation (A3) is equivalent to

$$\eth^n(\text{Þ}'_n\alpha) = -(n-1)!\,L_a\text{Þ}'\eta_n^0. \tag{A4}$$

We introduce a null tetrad $(N_a, L_a, M_a, \tilde{M}_a)$ on H defined by

$$N_a = \iota_A\iota_{A'} \qquad L_a = o_A o_{A'} \qquad M_a = o_A\iota_{A'} \qquad \tilde{M}_a = \iota_A o_{A'}.$$

It can easily be shown that, in the case of flat H space, these vectors satisfy

$$\begin{aligned} &\eth^2 L_a = 0, \qquad \eth^2 N_a = 0, \qquad \eth M_a = 0, \qquad \eth^3\tilde{M}_a = 0\\ &\eth\tilde{M}_a = N_a - L_a, \qquad \eth^2\tilde{M}_a = -2M_a\\ &\eth N_a = -M_a, \qquad \eth L_a = M_a. \end{aligned} \tag{A5}$$

Expanding $\eth'_a\alpha$ in terms of this tetrad, we have

$$\text{Þ}'_a\alpha = \beta N_a + \gamma L_a + \delta M_a + \epsilon M_a, \tag{A6}$$

where

$$\beta = L^a\text{Þ}'_a\alpha. \tag{A7}$$

Substituting (A6) into (A4) and using (A5), we obtain, after a little algebra, the relations

$$\eth^{n+1}\beta = 0, \tag{A8}$$

$$(n+1)/n\,\eth^n\beta - \eth^{n+1}\delta/n = -(n-1)!\,\text{Þ}'\eta_n^0. \tag{A9}$$

Equation (A9) implies that

$$\int\left(\frac{n+1}{n}\eth^n\beta - \frac{\eth^{n+1}\delta}{n}\right)o_{A'}\dots o_{K'}\,\mathrm{d}S = -(n-1)!\int \text{Þ}'\eta_n^0 o_{A'}\dots o_{K'}\,\mathrm{d}S$$

$$= \frac{n+1}{n}\int \eth^n\beta o_{A'}\dots o_{K'}\,\mathrm{d}S, \tag{A10}$$

where we have integrated by parts and used the fact that $\eth^{n+1}(o_{A'}\dots o_{K'})$ vanishes.

Substituting (A1) into (A10), we obtain

$$\eta_{A'\ldots K'} = \frac{n+1}{n!} \int \eth^n \beta o_{A'} \ldots o_{K'} \,\mathrm{d}S \tag{A11}$$

Consider now the spinor $\phi_{A'\ldots K'}$ defined by

$$\phi_{A'\ldots K'} = \frac{\eth^n \beta}{n!} o_{A'} \ldots o_{K'} - \frac{\eth^{n-1}\beta}{(n-1)!} o_{(A'} \ldots \iota_{K')} + \ldots + \beta \iota_{A'} \ldots \iota_{K'}. \tag{A12}$$

It can be checked that $\phi_{A'\ldots K'}$ has zero weight, and, using equation (A8), that it is independent of ζ and hence also of $\tilde{\zeta}$. Since $\int \mathrm{d}S = 1$, (A12) gives

$$\phi_{A'\ldots K'} = \frac{n+1}{n!} \int \eth^n \beta o_{A'} \ldots o_{K'}, \tag{A13}$$

where we have integrated by parts. Comparing (A13) with (A11), we see that $\phi_{A'\ldots K'}$ and $\eta_{A'\ldots K'}$ are identical.

We have therefore shown that, given β which has been obtained from the null datum η_n^0 of a spin-$n/2$ field by equations (A2) and (A7), the field is given by equation (A12).

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